



Scientific Background on the Nobel Prize in Physics 2016

TOPOLOGICAL PHASE TRANSITIONS AND
TOPOLOGICAL PHASES OF MATTER

compiled by the Class for Physics of the Royal Swedish Academy of Sciences

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1 Introduction

In 1972 J. Michael Kosterlitz and David J. Thouless identified a completely new type of phase transition in two-dimensional systems where topological defects play a crucial role [35, 36]. Their theory applied to certain kinds of magnets and to superconducting and superfluid films, and has also been very important for understanding the quantum theory of one-dimensional systems at very low temperatures.

In the early 1980s David J. Thouless and F. Duncan M. Haldane developed theoretical methods to describe phases of matter that cannot be identified by their pattern of symmetry breaking. In a 1982 paper, David Thouless and his collaborators Mahito Kohmoto, Peter Nightingale, and Marcel den Nijs, explained the very precise quantization of the Hall conductance in two-dimensional electron gases using topological concepts [51]. In 1983 Duncan Haldane derived a theory for spin chains that incorporated effects of topology in a crucial way [21, 23]. Based on this he predicted that chains with integer and half-integer spins should be qualitatively different, and this totally unexpected effect was later confirmed by experiments.

In the following, we shall first provide some background material to put the achievement of this year's Laureates in context, and then describe the discoveries themselves.

2 Background

Crystalline solids are a very important class of materials in which the atoms are arranged in periodic patterns. These patterns can be classified by their symmetries; the science of crystallography, based on observations of macroscopic crystals, predated the X-ray diffraction studies that allowed the positions of the atoms to be mapped in detail. The latter was a revolutionary development that was rewarded with two consecutive Nobel Prizes, in 1914 to Martin von Laue and in 1915 to William and Lawrence Bragg.

When a liquid solidifies into a crystal, it changes from a phase which is, on macroscopic scales, invariant under both translations and rotations, to a phase where these continuous symmetries are broken down to a finite symmetry group characteristic of the crystal. Another example of such a *phase transition* occurs when a ferromagnet is cooled below the Curie temperature, and the atomic magnetic moments, or the spins, line up and give rise to a net magnetization. This is illustrated in Fig. 1.

The study of magnetism has been very important for our understanding of

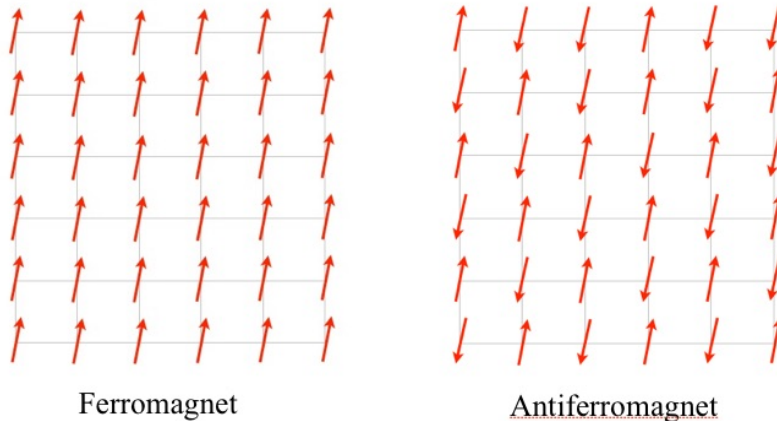


Figure 1: *Schematic picture of ferro- and antiferromagnets. The chequerboard pattern in the antiferromagnet is called a Néel state.*

the role of symmetry in physics. Using new experimental techniques, hidden patterns of symmetry were discovered. For example, there are magnetic materials where the moments form a chequerboard pattern where the neighbouring moments are anti-parallel, see Fig. 1. In spite of not having any net magnetization, such *antiferromagnets* are nevertheless ordered states, and the pattern of microscopic spins can be revealed by neutron scattering. The antiferromagnetic order can again be understood in terms of the associated symmetry breaking.

In a mathematical description of ferromagnetism, the important variable is the magnetization, $\vec{m}_i = \mu \vec{S}_i$, where μ is the magnetic moment and \vec{S}_i the spin on site i . In an *ordered phase*, the average value of all the spins is different from zero, $\langle \vec{m}_i \rangle \neq 0$. The magnetization is an example of an *order parameter*, which is a quantity that has a non-zero average in the ordered phase. In a crystal it is natural to think of the sites as just the atomic positions, but more generally one can define “block spins” which are averages of spins on many neighbouring atoms. The “renormalization group” techniques used to understand the theory of such aggregate spins are crucial for understanding phase transitions, and resulted in a Nobel Prize for Ken Wilson in 1982.

It is instructive to consider a simple model, introduced by Heisenberg, that describes both ferro- and antiferromagnets. The Hamiltonian is

$$H_F = -J \sum_{\langle ij \rangle} \vec{S}_i \cdot \vec{S}_j - \mu \sum_i \vec{B} \cdot \vec{S}_i \quad (1)$$

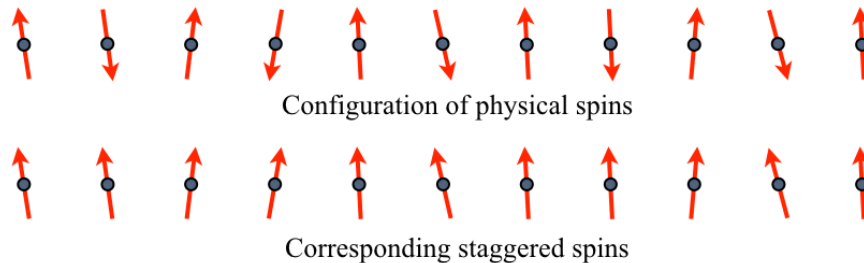


Figure 2: *Illustration of the concept of staggered magnetization. The upper figure shows the physical spins in an antiferromagnetic state, while the equivalent staggered spins in the lower figure are ferromagnetically ordered.*

where the spins are defined on lattice sites i and $\langle ij \rangle$ denotes nearest neighbours. The constant J determines the strength of the magnetic interaction, and μ is the magnetic moment of the atoms. If $J > 0$ the energy is lowest when all the spins are aligned in the direction of the external magnetic field \vec{B} , which explicitly breaks the symmetry under rotations and singles out a direction. When the magnetic field is zero, the spins are still aligned, but in an arbitrary direction. So in spite of the model being isotropic, the lowest energy state is not - the rotational symmetry is *spontaneously broken*.

Taking $J < 0$ favors the “chequerboard” *Néel state*, named after Louis Néel (Nobel Prize 1970), shown in the right panel of Fig. 1, so in this case Eq. (1) describes an antiferromagnet where the order parameter is the *staggered magnetization*, which is defined so that it is constant in the Néel state. In a chain, this amounts to defining $\vec{m}_i = \mu(-1)^i \vec{S}_i$ where the integer i numbers the sites on the chain as illustrated in Fig. 2.

Note that in spite of ferromagnetism and antiferromagnetism being quantum mechanical effects at the atomic level, they can nevertheless be described by the classical model in Eq. (1).

However, it was realized rather early on that at low temperatures there are macroscopic effects, both in liquid helium and in ordinary metals, that can not be understood in terms of classical physics. The discovery of superconductivity by Kamerlingh Onnes (Nobel Prize 1913) and of superfluid helium II Pyotr by Kapitsa (Nobel Prize 1978) firmly established the existence of superfluid phases of matter. The common feature of these phases is “condensation” which is most simply understood in a gas of non-interacting bosons. Here one can show that below the *Bose-Einstein condensation* temperature, a macroscopic number of particles will populate the lowest quantum mechanical energy eigenstate, and

these particles form a *condensate*. The case of the superconductor is somewhat more complicated. What condenses here are bosonic *Cooper pairs* of electrons, as explained by the BCS theory of superconductivity (Nobel Prize 1972 to John Bardeen, Leon Cooper and John Schrieffer).

These condensates can also be described by an order parameter which, in a sense, can be thought of as a “macroscopic wave function”, ψ , for the bosons, or for the Cooper pairs. In 1950, well before the advent of the microscopic BCS theory, Ginzburg and Landau proposed a theory for the order parameter that describes the phase transitions between the normal and superconducting phases, and some ten years later, the corresponding theory for the normal to superfluid transition in a gas of bosons, was given by Gross and Pitaevskii.

Although the order parameter ψ for a superfluid is a classical variable, it differs in a crucial way from the order parameter \vec{m} for a magnet, in that it is a complex number, where the phase of this complex number is a memory of its quantum mechanical origin. A phase ordered state, or a condensate, now amounts to having $\langle \psi \rangle \neq 0$ which means that the phase is constant, or slowly varying, in the whole system. This property is often referred to as *phase rigidity*. Since the phase of a quantum mechanical wave function is related to currents, variations in the phase of the order parameters correspond to “supercurrents” that flow without any resistance.

An important part of the Ginzburg-Landau (GL) theory is the potential function for the order parameter

$$V(\rho) = -\frac{\mu}{2}|\psi|^2 + \frac{\lambda}{4}|\psi|^4, \quad (2)$$

where μ is the chemical potential, or the cost of creating a Cooper pair, and where λ is the strength of the short range repulsion between the pairs. In GL theory these are both phenomenological parameters. Depending on whether μ is negative or positive, the potential will have a minimum either at $|\psi| = 0$ or at $|\psi| = \sqrt{\mu/\lambda}$. Thus, if μ depends on the temperature such that it becomes positive below some *critical temperature* T_c , this will correspond to a phase transition into a superconducting state.

A very important insight about superconductors was due to Abrikosov, who studied in detail the response of a GL superconductor to a magnetic field. It was already known from the work of Meissner and Ochsenfeld that superconductivity does not mix well with magnetism. When a material such as lead is placed in a magnetic field, and then is cooled below T_c , the magnetic field is expelled from the bulk of the material and only penetrates a thin region of depth λ_L close to the surface [18]. This is called the *Meissner effect*, and the *London length* λ_L is typically 50 - 500 nm. A superconductor with this

response to magnetic fields is said to be of type I. What Abrikosov found was that not all superconductors act like this. In a “type II” superconductor a sufficiently weak field will penetrate the material, but not as a homogeneous field; instead it is in the form of *Abrikosov vortices*, which are thin magnetic flux tubes with a diameter $\sim \lambda_L$. Only strong magnetic fields are totally expelled and a full Meissner effect is recovered. Alexei Abrikosov, Vitaly Ginzburg and Anthony Leggett shared the 2003 Nobel Prize for their work on superconductivity and superfluidity.

There is a striking similarity between a two-dimensional magnet and a superconducting or superfluid film. The magnetization is a vector that normally can point in any direction but in certain magnets, the spins are constrained to lie in a plane, say the xy-plane, where they are free to rotate. In such an “easy-plane” magnet the direction of the magnetization is determined by a single angle, θ denoting the rotation around the z-axis. It will be important in the following that there are configurations of such planar, or XY, spins that are *topologically distinct*. This is illustrated in the left panel of Fig. 3, which shows a vortex configuration, that cannot be transformed into the ground state where all the spins are aligned, by a continuous rotations of the spins.¹

The right panel shows a vortex anti-vortex configuration, which *can* be smoothly transformed to the ground state. In the next section we shall expand on this and classify the configurations by their vorticity, which is a *topological invariant*.

The complex order parameter for a superconductor or superfluid can be expressed as, $\psi = \sqrt{\rho_s} e^{i\theta}$, where ρ_s is the *superfluid density* and θ the phase. In the superconducting state, the important thermal fluctuations are only in the phase and are hence again described by a single angle, θ , just as in the easy plane magnet. An important concept in the general theory of phase transitions is that of *universality class*. What determines the universality class of a transition is the dimension of the system and the nature of the order parameter. Since the planar magnet and a superfluid both have order parameters described by a single angle, they belong to the same class and can be described by the same effective theory.

A simple model that describes both these systems is the XY-model defined

¹We assume that the lattice spacing is small so it makes sense to use a continuum description, where the lattice site is replaced by a position \vec{r} . With a continuous rotation we mean a transformation $\vec{S}(\vec{r}) \rightarrow \vec{S}'(\vec{r}) = \mathcal{R}(\vec{r})\vec{S}(\vec{r})$ where the matrix $\mathcal{R}(\vec{r})$ is a continuous function of \vec{r} .

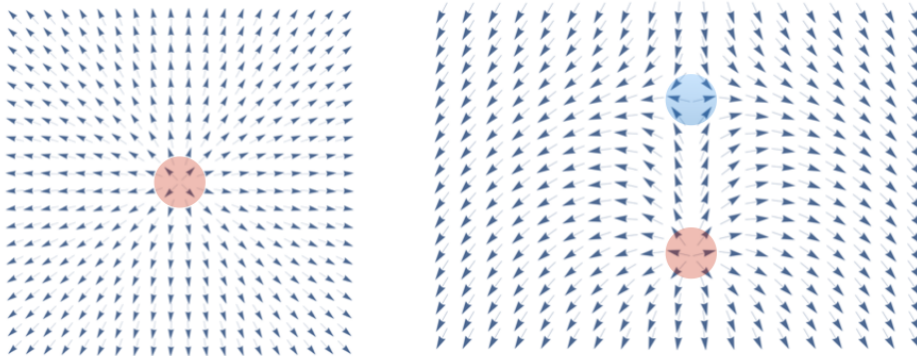


Figure 3: *To the left a single vortex configuration, and to the right a vortex-antivortex pair. The angle θ is shown as the direction of the arrows, and the cores of the vortex and antivortex are shaded in red and blue respectively. Note how the arrows rotate as you follow a contour around a vortex.*

by the Hamiltonian,

$$H_{XY} = -J \sum_{\langle ij \rangle} \cos(\theta_i - \theta_j) \quad (3)$$

where $\langle ij \rangle$ again denotes nearest neighbours and the angular variables, $0 \leq \theta_i < 2\pi$ can denote either the direction of an XY-spin or the phase of a superfluid. We shall discuss this model in some detail below.

Although the GL and BCS theories were very successful in describing many aspects of superconductors, as were the theories developed by Lev Landau (Nobel Prize 1962), Nikolay Bogoliubov, Richard Feynman and others for the Bose superfluids, not everything fit neatly into the Landau paradigm of order parameters and spontaneous symmetry breaking. Problems occur in low-dimensional systems, such as thin films or thin wires. Here, the thermal fluctuations become much more important and often prevent ordering even at zero temperature [39]. The exact result of interest here is due to Wegner, who showed that there cannot be any spontaneous symmetry breaking in the XY-model at finite temperature [53].

So far we have discussed phenomena that can be understood using classical concepts, at least as long as one accepts that superfluids are characterised by a complex phase. There are however important macroscopic phenomena that cannot be explained without using quantum mechanics. To find the ground state of a quantum many-body problem is usually very difficult, but there are some important examples where solutions to simplified problems give deep physical insights. Electromagnetic response in crystalline materials is an

example that is of central importance for this year's Nobel Prize.

That certain crystals are metals, and others are insulators can often be understood from the solution of the Schrödinger equation for a single electron in a periodic lattice potential. The crucial simplification neglects the electron-electron interaction, so that the ground state is obtained simply by filling the lowest energy levels. The important result is that the energy spectrum is not continuous, but forms bands of allowed energies with forbidden gaps in between. In an insulator, the itinerant electrons completely fill a number of bands and it takes significant energy to generate a current. In a metal, the highest populated band is only partially filled, and there are low energy excitations that allow for conduction.

This simple picture was challenged by the 1980 discovery by Klaus von Klitzing (Nobel Prize 1985) of the integer quantum Hall effect. This was the first discovery of a topological quantum liquid, with many more to come, and it demonstrated that band theory was much richer than expected.

The classical example of a phase transition is a system going from a disordered phase to an ordered phase as the temperature is lowered below a critical value. More recently, the phase transition concept has been extended to quantum systems at zero temperature. A quantum system can undergo a radical change of its ground-state as a parameter in its Hamiltonian, such as pressure, magnetic field or impurity concentration, is tuned through a critical value, and such a *quantum phase transition* signals the change from one state of matter to another. This insight has provided an important link between statistical mechanics, quantum many-body physics and high energy physics, and these fields now share a large body of theoretical techniques and results.

3 The Kosterlitz-Thouless phase transition

We already mentioned that the thermal fluctuations prevent ordering of XY-spins in two dimensions. A more precise statement is based on the large distance behaviour of the spin-spin correlation function. In three dimensions a calculation gives,

$$\lim_{r \rightarrow \infty} \langle e^{i(\theta(\vec{r}) - \theta(\vec{0}))} \rangle = \begin{cases} c_1 & T < T_c \\ c_2 e^{-r/\xi} & T > T_c \end{cases}$$

where ξ is the correlation length, $r = |\vec{r}|$, and c_1 and c_2 are constants. At precisely the critical temperature $T = T_c$ the correlation falls as a power *i.e.* $\sim r^{-(1+\eta)}$, signalling a *critical behaviour*. The constant η is an example of

a critical exponent which is characteristic of a *universality class*, which can encompass many different systems that all behave in a similar way close to the phase transition.

To see what happens in two dimensions, we take the continuum limit of the Hamiltonian Eq. (3), to get

$$H_{XY} = \frac{J}{2} \int d^2r (\vec{\nabla}\theta(\vec{r}))^2. \quad (4)$$

A simplification is to extend the range of the angular variable to $-\infty < \theta < \infty$ to get a free field Hamiltonian and thus Gaussian fluctuations, and a direct calculation using a short distance cutoff a gives

$$\langle e^{i(\theta(\vec{r})-\theta(\vec{0}))} \rangle \sim \left(\frac{a}{r}\right)^{\frac{k_B T}{2\pi J}}. \quad (5)$$

This is a power law even at high temperatures, where an exponential fall-off would be expected. Kosterlitz and Thouless [35, 36] resolved the apparent paradox by showing that there is indeed a finite temperature phase transition, but of a new and unexpected nature where the vortex configurations play an essential role.

The glitch in the argument leading to Eq. (5) is that the periodic, or U(1), nature of θ cannot be ignored, since that amounts to neglecting vortex configurations. A vortex like the one in Fig. 3 is characterised by a non zero value of the vorticity,

$$v = \frac{1}{2\pi} \oint_C d\vec{r} \cdot \vec{\nabla}\theta(\vec{r}) \quad (6)$$

where C is any curve enclosing the centre position of the vortex. The integral measures the total rotation of the spin vector along the curve, so after dividing with 2π , v is simply the number of full turns it makes when circling the vortex. From this, we also understand that there can also be antivortices, where the spin rotates in the opposite direction as seen in the right panel in Fig.3. For a rotationally symmetrical vortex with $v = \pm 1$ it follows from Eq. (6) that $|\vec{\nabla}\theta(\vec{r})| = 1/r$, so the energy cost for a single vortex becomes,

$$E_v = \frac{J}{2} \int d^2r \left(\frac{1}{r}\right)^2 = J\pi \ln \frac{L}{a} \quad (7)$$

where L is the size of the system, and a a short distance cutoff that can be thought of as the size of the vortex core. So for a large system, the energy cost

for a single vortex is very large, and cannot be excited by thermal fluctuations. This seems to imply that vortices can be neglected, but we shall see that this is not the case.

One year before the work of Kosterlitz and Thouless, Vadim Berezinskii (died in 1981) also recognized the importance of vortex excitations in the XY-model [8, 9], but did not conclude that they could give rise to a phase transition at finite temperature. Therefore, in the following we shall use “KT-transition”.

We can understand the essence of this new type of *topological phase transition* by a quite simple thermodynamic argument. Although the energy of a single vortex diverges as $\ln L$, this is not true for vortex-antivortex pairs since they have zero total vorticity. The energy required to create such a pair is $J2\pi \ln r/a$ where r is the separation between the vortices. Such pairs can thus be thermally excited, and the low temperature phase will host a gas of such pairs. The insight by Kosterlitz and Thouless was that at a certain temperature T_{KT} the pairs will break up into individual vortices. It is this vortex pair unbinding transition that will take the system to a high temperature phase with exponentially decaying correlations.

The vortices and anti-vortices act as if they were two point particles with charges $+1$ and -1 interacting with a $1/r$ force. Since this corresponds to the Coulomb interaction in two dimensions, the physics of the topological defects is just like the physics of a two-dimensional neutral Coulomb gas. Thouless’ and Kosterlitz’s heuristic entropy-energy balance argument for the unbinding transition is as follows: The free energy for a single vortex is

$$F = E - TS = J\pi \ln \left(\frac{L}{a} \right) - Tk_B \ln \left(\frac{L^2}{a^2} \right) \quad (8)$$

where k_B is Boltzmann’s constant, and where the entropy is calculated assuming that there are L^2/a^2 possible positions for a vortex with area a^2 . At the critical temperature $T_{KT} = J\pi/2k_B$ the energy exactly balances the entropy, so we can expect the transition to a phase of free vortices.

In contrast to usual continuous phase transitions, the KT-transition does not break any symmetry, something that was completely new and unexpected. In their 1973 paper [36] Kosterlitz and Thouless both explained the physics behind the transition and fully recognized its importance. The next important contribution to the theory was Kosterlitz’ derivation of the *Kosterlitz renormalization group equations* and his analysis of the associated flow [34].

The Kosterlitz-Thouless topological model of a phase transition in two dimensions has been used to explain experiments with many different types of

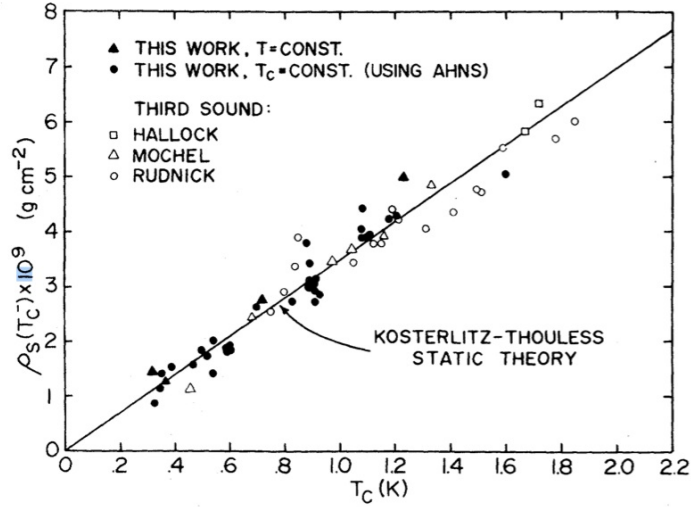


Figure 4: At the Kosterlitz-Thouless transition the superfluid density and critical temperature are predicted to have a linear relation depending only on fundamental constants $\rho(T_c) = T_c \frac{2}{\pi} \frac{m^2 k_B}{\hbar^2}$. (Figure from Ref. [10].)

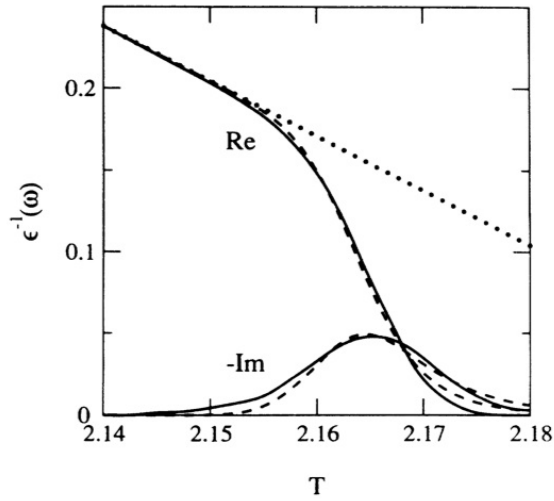


Figure 5: Solid lines: The measured real and imaginary parts of the linear response an AC circuit containing a two-dimensional wire network of superconductors as it is cooled through the Kosterlitz-Thouless transition (data from [28]). Dashed lines: The theory of Minnhagen [40] is fit to the data. A narrow temperature range reveals a drop in the superfluid density $\sim \text{Re}(\epsilon(\omega))$ and a peak in the dissipation $\sim \text{Im}(\epsilon(\omega))$ caused by vortex-unbinding around the transition temperature. (Figure from Ref. [52].)

physical systems. Examples include: very thin films of superfluid ^4He that form naturally on a solid substrate [10], disordered thin films of superconductors [17], artificial planar arrays of superconducting tunnel junctions [47] and wire networks [38], granular films of superconductors [26], melting of two-dimensional solids [50].

An important idea due to Nelson and Kosterlitz [43] that is used in the analysis of superfluid and superconducting films is that of a “universal jump” in the superfluid density, which occurs at the critical temperature of the phase transition. At the critical temperature, the superfluid density jumps from zero to a universal value predicted by the Kosterlitz-Thouless theory. Bishop and Reppy [10] measured this jump by monitoring the resonant frequency and quality factor of a torsional oscillator with a large spiral surface of Mylar plastic. When the system is cooled below the critical temperature, a thin superfluid film forms on the Mylar substrate and the added mass of this film causes a sudden jump in the period of torsional oscillation. Bishop and Reppy compared their results with other experiments by Rudnick, as well as by Hallock and by Mochel, which measured the temperature dependence of the velocity of a particular “third sound” surface wave in liquid helium. See Fig. 4. All results were consistent with the prediction of a universal jump at the KT-transition.

Beasley, Mooij and Orlando [7] predicted that the KT-transition would also be visible in thin films of superconductors if they were made to be sufficiently disordered. The jump in the density of superconducting electrons appears in analysis of the nonlinear current-voltage characteristics of the thin superconducting film as it is cooled through the superconducting transition. Early experiments on thin disordered superconducting films [17] were consistent with this picture. Some of the most elegant experiments and analysis of superconducting systems detect the phase transition as a change in the complex impedance of the two-dimensional superconductor as it is cooled through the KT-transition [28, 52]. See Fig. 5.

4 Quantum Hall conductance and topological band theory

The discovery of the integer quantum Hall effect was a milestone in the understanding of the phases of matter. The Hall conductance in a very clean (mobility about $10^5 \text{ cm}^2/\text{Vs}$) two-dimensional electron gas cooled below 2 K and subjected to a perpendicular magnetic field of the order of 15 T, was

observed to obey the relation

$$\sigma_H = n \frac{e^2}{h}, \quad (9)$$

where n is an integer, to a precision of less than one part in 10^9 . This finding was simply astonishing, given the much larger relative variations in magnetic field, impurity concentration and temperature.²

Using an ingenious thought experiment, Robert Laughlin (Nobel Prize for the fractional quantum Hall effect in 1998) gave an argument based on gauge invariance to explain the exactness of the Hall conductance [37]. However, deeper understanding of the Hall response in real crystalline materials was missing, and a straightforward application of Laughlin’s reasoning led to an apparent paradox. In resolving these problems, Thouless *et al.* derived a new formula for σ_H that turned out to have far reaching consequences [51]. Here we present a much simplified version of their argument that stresses the relevance of topology.

If we neglect electron–electron interactions, and replace the lattice potential with a smeared-out positive background charge, we only have charged particles moving independently in a magnetic field. This problem was solved by Landau, who showed that the energy is quantized as $E_n = (n + \frac{1}{2})\hbar\omega_c$ with the cyclotron frequency $\omega_c = eB/m$. Each such “Landau level” has a macroscopic degeneracy such that there is one quantum state for every unit $\phi_0 = h/e = 2\pi/e$ of magnetic flux. There are many ways to label the degenerate states and a convenient one for our purposes is to use a crystal momentum. The reason for this is that even in a homogenous magnetic field the symmetry under translations is broken, and the relevant symmetry group is that of finite “magnetic translations”. These translations span a lattice such that there is a unit magnetic flux through each lattice cell. The particular expressions for the magnetic translation operators are not important for our discussion, nor is the precise choice of the lattice. What is important, however, is that just as for an ordinary crystal lattice, we can label the eigenfunctions in a band, here the Landau level, with a vector in reciprocal space. We say that the crystal momentum, \vec{k} , takes values in the first Brillouin zone, and we denote the wave functions in the n^{th} Landau level by $u_{\vec{k},n}(\vec{r})$.

It is now an exercise in linear response theory to derive an expression for

²The measurement is so precise that the Klitzing constant, $R_K = h/e^2 = 25812.807557(18)\Omega$ is now the definition of electrical resistance.

the conductance in terms of the wave functions; the final formula becomes

$$\sigma_H = \frac{e^2}{2\pi h} \sum_n \int_{\vec{k} \in BZ} d^2k \mathcal{B}(\vec{k}, n) \quad (10)$$

where the *Berry field strength*, \mathcal{B} can be calculated from the *Berry potential*, which is given in terms of the wave functions by

$$\mathcal{A}_j(\vec{k}, n) = i \langle u_{\vec{k}, n} | \partial_{k_j} | u_{\vec{k}, n} \rangle. \quad (11)$$

Just as in ordinary electromagnetism, the field strength is related to the vector potential by,

$$\mathcal{B}(\vec{k}, n) = \partial_{k_x} \mathcal{A}_y(\vec{k}, n) - \partial_{k_y} \mathcal{A}_x(\vec{k}, n). \quad (12)$$

The integral in Eq. (10) is of a (fictitious) magnetic field over a closed surface, so, in analogy with the case of a magnetic monopole in ordinary electromagnetism, it is expected to be quantized. A detailed analysis in fact shows that,

$$\frac{1}{2\pi} \int_{Bz} \mathcal{B}(\vec{k}, n) = C_1(n), \quad (13)$$

where the first *Chern number*, C_1 , is known from the mathematics of fibre bundles to be an integer. This explains why the conductance is quantized, and why it is insensitive to perturbations such as disorder or interactions between particles – the Chern number is a *topological invariant*. It can only be an integer! Using the explicit wave functions for the Landau problem, one can explicitly calculate the integrals in Eq. (10), and show that $C_1 = 1$ for any filled Landau level.

The original paper by Thouless *et al.* also included the effect of a lattice potential. The calculations become more complicated, since in that case the Landau levels split into subbands, but the result is the same - the conductance is still given by the formula Eq. (9). In later papers, the relation to the mathematics of fibre bundles was established in Refs. [5] and [33], and in Ref. [44] Niu, Thouless and Wu gave a derivation that also applied to systems with impurities.

The great importance of the Thouless *et al.* result is that it opens up the possibility of having a Hall conductance even in the absence of a magnetic field. It would, however, be another six years before this conceptually very important step was taken by Haldane [22], who realized that the important thing is to break the invariance under time-reversal and have an energy band,

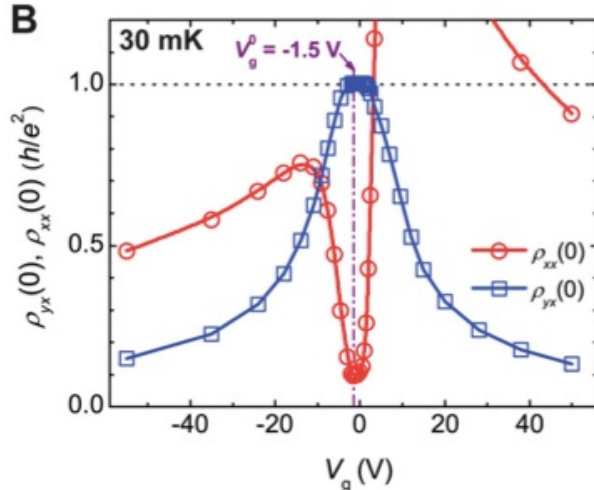


Figure 6: The blue curve shows the Hall resistance ρ_{yx} as a function of the gate voltage at zero magnetic field. Note the plateau at $V_g = 0$, which is the point corresponding to a filled band. (Figure from Ref. [15].)

which does not have to be a Landau level, with a non-zero Chern number. He considered a tight-binding model of fermions on a hexagonal lattice and introduced hopping between both nearest and next-nearest neighbours.

In a lattice model, magnetic flux is incorporated by making the hopping matrix elements complex, and Haldane picked the phases to give fluxes with alternating signs giving zero flux in each unit cell. This configuration still breaks the invariance under time reversal, which is necessary to have a Hall effect. This phase of matter described by Haldane is now called a *Chern insulator*, and twentyfive years later, in 2013, a quantized Hall effect was observed in thin films of Cr-doped $(\text{Bi,Sb})_2\text{Te}_3$ at zero magnetic field, thus providing the first experimental detection of this phase of matter [15]. In Fig. 6 we see a clear plateau in the Hall resistance ρ_{yx} at a density (regulated by the gate voltage) corresponding to a filled band. The later development of *topological band theory* will be discussed in the concluding section.

5 Quantum spin chains and symmetry-protected topological phases of matter

One dimensional systems, such as spin chains, or electrons moving in thin wires, are radically different from their relatives in higher dimensions. The

reason for this is that both thermal and quantum fluctuations are much more important and prevent most of the symmetry-breaking patterns that characterise phases in higher dimension. A lot of important work in the 1960s and 1970s had established quite a complete and coherent picture of both quantum and classical one-dimensional systems. In the quantum case there are various transformations, both in the continuum and on the lattice, that map fermionic systems to bosonic ones, such as the Jordan–Wigner transformation that maps the Heisenberg chain of spin 1/2’s to (spin less) lattice fermions, with nearest neighbour interactions. Since the spin chain can be solved exactly using the Bethe ansatz techniques, this also provides a solution for the fermion model, which is but an example of the many cross-connections between different one-dimensional models.

The antiferromagnetic Heisenberg chain, illustrated in Fig.2, is for spin 1/2 described by Eq.(1) with $\vec{S}_i = \frac{\hbar}{2}\vec{\sigma}_i$ (with $\vec{\sigma}$ the Pauli matrices), and the Bethe ansatz solution shows that it is gapless. Although there were no proofs, it was commonly believed that the same would be true for Heisenberg chains of higher spins. In two papers from 1983 [21, 23], Haldane applied new mathematical techniques to the problem, revealing a fundamental difference between chains with integer and half integer spins, leading him to the famous “Haldane conjecture” that half-integer chains are gapless while the integer ones are gapped.

The key idea was to derive an effective model that describes the low momentum excitations.³ Assuming the spins to be large, Haldane derived the following action integral for the continuum limit of the antiferromagnetic spin chain,

$$S_{NLS} = \frac{1}{2g} \int dt dx \left(\frac{1}{v} (\partial_t \vec{n})^2 - v (\partial_x \vec{n})^2 \right) \quad (14)$$

where \vec{n} is a unit vector describing the slowly varying part of the staggered spin field, v is the spin wave velocity and $g = 2/S$ the coupling constant. This is the $O(3)$ non-linear sigma model, which at the time was already well understood. Although naively the theory has no mass, it was known that because of strong quantum fluctuations a mass scale is dynamically generated⁴ [46] so, according to this line of argument, all spin chains should be gapped. This is in apparent

³How this is done in detail is well explained in the textbooks Ref. [4, 19].

⁴Note that large S corresponds to a small coupling g , which effectively suppresses the contributions to the path integral from configurations with large fluctuations. A renormalization group analysis shows that the theory is *asymptotically free*, meaning that the coupling constant grows at small momenta, very similar to QCD. It is these strong quantum fluctuations that drive the mass generation.

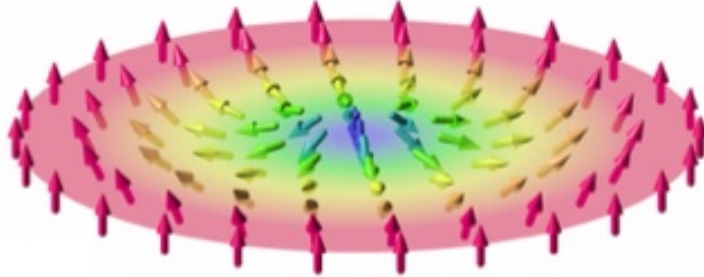


Figure 7: A field configuration with winding number $Q = 1$. Notice that no continuous change in the spin directions can transform this to a configuration where all spins point upwards. (Figure by Karin Everschor-Sitte and Matthias Sitte).

contradiction to the spin 1/2 chain being gapless. Haldane pointed out that there are large fluctuations that contribute very differently depending on the value of the spin.

One way to understand this difference is to notice that a direct derivation of the action integral will, in addition to Eq. (14), also give a *topological θ -term*

$$S_{top} = i \frac{\theta}{4\pi} \int d^2x \vec{n} \cdot (\partial_1 \vec{n} \times \partial_2 \vec{n}), \quad (15)$$

where $\theta = 2\pi S$, and where we use the Euclidean space coordinates $(x^1, x^2) = (it, x)$ appropriate for a path integral treatment.⁵ Although this term does not contribute to the equations of motion, it is nevertheless important. To see this we first notice that, for any smooth field configuration, the *winding number*

$$Q = \frac{1}{4\pi} \int d^2x \vec{n} \cdot (\partial_1 \vec{n} \times \partial_2 \vec{n}), \quad (16)$$

is an integer. The geometric significance of this winding number is explained in Fig. 7.

In a path integral where one sums over all possible spin configurations to calculate various quantities, such as the partition function,

$$Z(g) = \int \mathcal{D}[\vec{n}(\vec{x})] e^{-(S_{NLS} + S_{top})} \quad (17)$$

⁵In the original work [23] another line of reasoning, also employing topological concepts, was used to reach the same conclusion. For the derivation of the θ -term and references to the original papers, see [19].

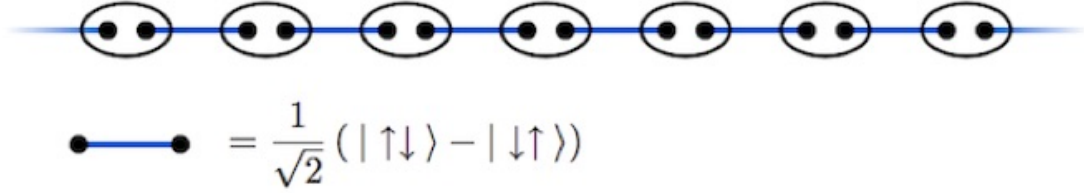


Figure 8: *Graphical representation of the AKLT ground state. The black dots denote auxiliary spin 1/2 “particles”, the ovals project on spin 1, and the lines mean that two spins 1/2 form a singlet state. Since two of the four spin 1/2 on two adjacent sites form a singlet, the maximal total spin is 1, so the projection on spin 2 gives zero. Since this is true for all pairs, this state is clearly an eigenstate of H_{AKLT} . Also note the unpaired spins at the end of the chain, which are the fractionalized edge modes. (Figure taken from the Wikipedia article: AKLT model.)*

there will be a phase factor $e^{i2\pi SQ}$. It follows that for integer spins this factor is always 1, and we conclude that the chain is gapped. For half-integer chains the problem is much more complicated. The large fluctuations responsible for generating the mass gap typically have non-zero winding numbers and, because of the sign, $(-1)^Q$, there may be important cancellations. Thus, although the argument based on the behaviour of the sigma model works for the integer spin chains, it breaks down in the half-integer case. This observation, together with the spin 1/2 chain being gapless, provides a motivation for the Haldane conjecture. Note that the most surprising result – that the integer spin chain is gapped – is natural in the language of the sigma model, while it was harder to understand what happens in the half-integer case. Only later was it proven that the $\theta = \pi$ sigma model really is gapless [48].

We now complement the above rather abstract argument, which also relied on the assumption of large S , with a description of a very instructive and exactly solvable model for $S = 1$, which is a close cousin of the Heisenberg model in Eq. (1). The Hamiltonian for this AKLT chain, named after its inventors Ian Affleck, Tom Kennedy, Elliott Lieb and Hal Tasaki, is [1]

$$H_{AKLT} = \sum_i \left[\frac{1}{2} \vec{S}_i \cdot \vec{S}_{i+1} + \frac{1}{3} (\vec{S}_i \cdot \vec{S}_{i+1})^2 \right] = \sum_i P_2(\vec{S}_i + \vec{S}_{i+1}) \quad (18)$$

where \vec{S}_i is a spin 1 operator at the lattice site i , and P_2 projects on the subspace corresponding to spin 2 on two adjacent lattice sites. To find the ground state, we imagine that each link in the chain hosts two auxiliary spin 1/2 that are projected to a spin 1. As explained in Fig. 8, forming a spin

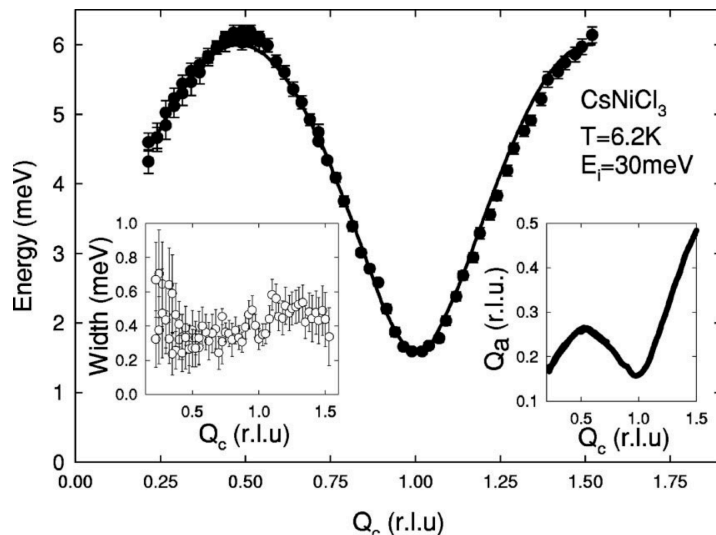


Figure 9: The graph in the middle shows the energy of a spin excitation in a spin 1 chain as a function of momenta close to the Néel point $Q_c = 1$, which corresponds to a π phase difference between the Ni spins along the chains; the Haldane gap is clearly visible. (Figure from Ref. [31].)

singlet at each link in the chain gives an eigenstate of the Hamiltonian with zero energy. Since the Hamiltonian is a sum of projectors, the ground state energy has to be non-negative and we conclude that we have constructed a ground state of the full interacting model. From the figure we also see that there are two “unpaired” spin 1/2 degrees of freedom at the two ends of the chain, which is an example of *quantum number fractionalization*, since the original degrees of freedom were spin 1! One can show that the unpaired spins give rise to a double degeneracy of the ground state, but the most striking property of the AKLT chain is that it has a *Haldane gap*, as was shown analytically in a later article by the same authors [2].

The existence of the *Haldane phase* has been confirmed both by experiments and by numerical simulations. The first experiment on CsNiCl₃ was done by Buyers *et al.* [13], and in Fig. 9 we show results from a more recent experiment [31].

Later work has greatly deepened our understanding of the Haldane phase of the Heisenberg antiferromagnetic chain. Although there is no local order parameter, it is sometimes possible to characterise it by a non-local *string order parameter* [30] introduced earlier in the context of statistical mechanics [16]. To define a distinct phase of matter, it is important that the characteristic

properties are not destroyed by small perturbations. For the Haldane phase this was investigated in 2010 by Pollmann *et al.*, who showed that as long as certain symmetry properties are respected by the perturbations, the phase remains intact and exhibits a characteristic entanglement property [45]. Thus the Haldane phase was identified as the charter member of a class often referred to as *symmetry protected topological states*, and we shall provide some other examples below.

6 Some recent developments

In the previous sections we described the fundamental work that is rewarded with this year’s Nobel Prize. We shall now try to give an impressionistic view of some of the exciting research directions and results that have followed.

6.1 What states of matter are there?

This is an ancient question. The simple schoolbook answer is solids, liquids and gases, but we know that this simple division is far too crude. We have already mentioned different kinds of magnets and superfluids, but there are many more examples. Most real materials are in fact much more complex than homogeneous crystals, liquids or gases; plasmas, liquid crystals polymers and gels are but a few examples. In spite of this enormous variety there are many important phenomena that can be understood in the simple context of crystalline materials with impurities and lattice defects. The theory of electronic bands explains why materials are conductors, isolators or semiconductors, and how they conduct heat and respond to magnetic fields. The discovery of the quantum Hall effect, and the subsequent development of topological band theory, has opened new and unexpected vistas where deep theoretical insights have developed in parallel with search for applications in electronics and quantum information science.

Perhaps the most spectacular results so far have been the predictions, and later experimental discoveries, of topological insulators in both two and three dimensions [24]. These states of matter are, just like the Haldane chain, examples of symmetry protected topological phases, as are the topological superconductors. Another example that has recently gained a lot of attention is the “Kitaev chain” [32], in its various incarnations. These are the simplest examples of materials harboring *Majorana modes*, which can be thought of as a fractionalized qubit. This observation has generated lots of excitement because of the possibility of topological quantum computation [42]. A more

recent, but equally fascinating set of materials are the Weyl semi metals [27], which were experimentally discovered as recently as 2015 [54].

There is now a topological classification of gapped phases of free fermions in any dimension, and many efforts are being made to classify interacting phases. Important tools in this effort to enumerate and classify phases of matter are different measures of quantum entanglement, such as entanglement entropy and entanglement spectra. One distinguishes between short-range entangled states, such as the symmetry protected states, the integer quantum Hall states and the Chern insulator, and states with long range entanglement such as the fractional Hall liquids or the putative *spin liquids* [6]. The long-range entangled states are characterised by having fractionalized excitations in the bulk, the typical example being the fractionally charged quasiparticles in the Laughlin quantum Hall liquids. The hunt for spin liquids, in both two and three dimensions, is very much at the forefront of current research, as is the attempt to realize states where the quasiparticles have nonabelian fractional statistics [41].

6.2 Quantum simulations, and artificial states of matter

In our discussion of the KT phase transition, we stressed universality, meaning that the same model Hamiltonian describes critical phenomena in very different physical systems. This universality is however to a very small *critical region* in the vicinity of the transition temperature. There is however another strategy that allows the use of the same Hamiltonian for different systems in wider parameter ranges. The basic idea goes back to Feynman, who pointed out that one could hope to solve very hard quantum problems by designing a *quantum simulator*.

Such a simulator is itself a quantum system with many degrees of freedom, but it should be well controlled and designed to embody the important aspects of the physical system one is attempting to simulate. More precisely, this means having the correct degrees of freedom and the correct interactions. Cold atomic gases have turned out to provide a perfect platform for obtaining this [11]. An important tool in these experiments is the optical lattices that are formed as an interference pattern by intersecting laser beams.

An example of this is the observation of the KT-transition in a layered Bose-Einstein condensate of ^{87}Ru atoms. At low temperatures one observes coherence effects characteristic of the phase with power law correlations, and at higher temperatures one sees free vortices [20].

We have already mentioned that the KT theory also describes the quantum phase transition in the one-dimensional XY universality class. Imaginary time

provides the extra dimension, and the control parameter analogous to temperature is the ratio of two energy scales in the Hamiltonian [12, 49]. In this way, the KT-transition forms the basis for understanding how a one-dimensional chain of Josephson tunnel junctions undergoes a zero-temperature transition from superconducting to insulating behaviour as the Josephson coupling between junctions is tuned [25]. The same model was later realized using ultra cold atomic gases trapped to form discrete lattices, and also here one could observe the KT-transition [14].

Both fermionic and bosonic atoms can be trapped in optical lattices, similarly to electrons in a crystal lattice. This also makes it possible to engineer topologically nontrivial bands, and we mention two recent examples.

In a 2014 experiment, the group led by Immanuel Bloch managed to design a lattice with topologically non-trivial bands, similar to the ones studied in the famous paper by Thouless *et al.* [51]. These bands were then populated by a gas of cold bosonic ^{87}Rb atoms, and using an intricate measuring procedure, they could experimentally determine the Chern number for the lowest band to $C_1^{exp} = 0.99(5)$ [3].

Also in 2014, the group led by Tilman Esslinger made an experiment with cold ^{40}K atoms in an optical lattice to simulate the precise model proposed by Haldane in 1988 [29]. This shows that reality sometimes surpasses dreams. At the end of his paper Haldane wrote: “While the particular model presented here, is unlikely to be directly physically realizable, it indicates ...”. What he could not imagine was that 25 years later, new experimental techniques would make it possible to create an *artificial state of matter* that would indeed provide that “unlikely” realization.

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